1.The Lagrange's Method

Consider choosing x_1 *and* x_2 *to maximise* $f(x_1, x_2)$ *, when* x_1 *and* x_2 *must satisfy some particular relation to each other that we write in implicit form as* $g(x_1, x_2) = 0$

We write this problem as follows

$$\max_{x_1, x_2} f(x_1, x_2) \quad \text{s.t.} \quad g(x_1, x_2) = 0$$

Suppose we multiply the constraint equation by a new variable, call it λ (lambda), we will have constructed a new function, called the Lagrangian function, or Lagrangian for short, and denoted by a script $L(\cdot)$.

$$L(x_1, x_2, \lambda) \equiv f(x_1, x_2) + \lambda g(x_1, x_2)$$

We would take all three of its partial derivatives and set them equal to zero

$$\frac{\partial L}{\partial x_1} = \frac{\partial f\left(x_1^*, x_2^*\right)}{\partial x_1} + \lambda^* \frac{\partial g\left(x_1^*, x_2^*\right)}{\partial x_1} = 0$$

$$\frac{\partial L}{\partial x_2} = \frac{\partial f\left(x_1^*, x_2^*\right)}{\partial x_2} + \lambda^* \frac{\partial g\left(x_1^*, x_2^*\right)}{\partial x_2} = 0$$

$$\frac{\partial L}{\partial \lambda} = g\left(x_1^*, x_2^*\right) = 0$$

Consider our contrived function $L(\cdot)$ *and take its total differential*

$$dL = \frac{\partial L}{\partial x_1} dx_1 + \frac{\partial L}{\partial x_2} dx_2 + \frac{\partial L}{\partial \lambda} d\lambda$$

By assumption, dL=0, so

$$dL = \frac{\partial f\left(x_1^*, x_2^*\right)}{\partial x_1} dx_1 + \frac{\partial f\left(x_1^*, x_2^*\right)}{\partial x_2} dx_2 + g\left(x_1^*, x_2^*\right) d\lambda$$

$$+ \lambda^* \left[\frac{\partial g\left(x_1^*, x_2^*\right)}{\partial x_1} dx_1 + \frac{\partial g\left(x_1^*, x_2^*\right)}{\partial x_2} dx_2 \right] = 0$$

The constraint is satisfied at x_1^* *and* x_2^**, so* $g\left(x_1^*, x_2^*\right) = 0$

We can identify those changes dx_1 *and* dx_2 *that make* $dg = 0$ *by totally differentiating the constraint equation and setting it equal to zero. So*

$$dg = \frac{\partial g\left(x_1^*, x_2^*\right)}{\partial x_1} dx_1 + \frac{\partial g\left(x_1^*, x_2^*\right)}{\partial x_2} dx_2 = 0$$

At $\left(x_1^*, x_2^*\right)$

$$dL = \frac{\partial f\left(x_1^*, x_2^*\right)}{\partial x_1} dx_1 + \frac{\partial f\left(x_1^*, x_2^*\right)}{\partial x_2} dx_2 = 0$$

This is to say that

$$\text{If } \left(x_1^*, x_2^*, \lambda^*\right) \text{ solves } dL\left(x_1^*, x_2^*, \lambda^*\right) = 0 \text{ for all } \left(dx_1, dx_2, d\lambda\right),$$

$$\text{then } df\left(x_1^*, x_2^*\right) = 0 \text{ for all } dx_1 \text{ and } dx_2$$

Nature:

The solution $\left(x_1^*, x_2^*\right)$ *in* $\left(x_1^*, x_2^*, \lambda^*\right)$ *that optimises L ,* *and* $\left(x_1^*, x_2^*\right)$ *that automatically optimises* $f\left(x_1, x_2\right)$*,* *satisfies* $\mathrm{g}\left(x_1, x_2\right) = 0$

Lagrange's method 'works' for functions with any number of variables, and in problems with any number of constraints, as long as the number of constraints is less than the number of variables being chosen. Suppose we have a function of n variables and we face m constraints, where m < n.

$$\max_{\mathbf{x}} f(\mathbf{x}) \quad \text{s.t.} \quad g^{j}(\mathbf{x}) = 0 \quad j = 1, \ldots, m$$

To solve this, form the Lagrangian by multiplying each constraint equation g^{j} *by a different Lagrangian multiplier* λ_j *and subtracting them all from the objective function* f.

$$L(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}) - \sum_{j=1}^{m} \lambda_j g^{j}(\mathbf{x})$$

The first-order conditions again require that all partial derivatives of L be equal to zero at the optimum. Because L has n + m variables, there will be a system of n + m equations determining the n + m variables $\mathbf{x}^*$ *and* $\boldsymbol{\lambda}^*$

$$\frac{\partial L}{\partial x_i} = \frac{\partial f\left(\mathbf{x}^*\right)}{\partial x_i} - \sum_{j=1}^{m} \lambda_j^* \frac{\partial g^j\left(\mathbf{x}^*\right)}{\partial x_i} = 0$$

$$\frac{\partial L}{\partial \lambda_j} = g^j\left(\mathbf{x}^*\right) = 0 \quad j = 1, \ldots, m$$