

4.CES Utility Function

CES utility function represents preferences that are strictly monotonic and strictly convex.

$$u(x_1, x_2) = (x_1^{\rho} + x_2^{\rho})^{1/\rho} \quad 0 \neq \rho < 1$$

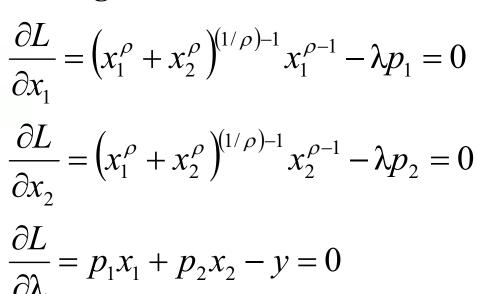
The consumer's problem is to find a nonnegative consumption bundle solving

$$\max_{x_1, x_2} (x_1^{\rho} + x_2^{\rho})^{1/\rho} \quad \text{s.t.} \quad p_1 x_1 + p_2 x_2 - y \le 0$$

To solve this problem, we first form the associated Lagrangian

$$L(x_1, x_2, \lambda) = (x_1^{\rho} + x_2^{\rho})^{1/\rho} - \lambda(p_1 x_1 + p_2 x_2 - y)$$

Assuming an interior solution



Then

$$x_{1} = x_{2} \left(\frac{p_{1}}{p_{2}}\right)^{1/(\rho-1)}$$
$$y = p_{1}x_{1} + p_{2}x_{2}$$



Solving for x_2 and x_1 gives the solution value



$$x_2 = \frac{p_2^{1/(\rho-1)}y}{p_1^{\rho/(\rho-1)} + p_2^{\rho/(\rho-1)}}$$

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Let
$$r = \rho/(\rho-1)$$

Then we can write the Marshallian demands as

$$x_1(\mathbf{p}, y) = \frac{p_1^{r-1}y}{p_1^r + p_2^r}$$

$$x_2(\mathbf{p}, y) = \frac{p_2^{r-1}y}{p_1^r + p_2^r}$$

By substituting the Marshallian demands back into the direct utility function, we get



$$v(\mathbf{p}, y) = \left[(x_1(\mathbf{p}, y))^{\rho} + (x_2(\mathbf{p}, y))^{\rho} \right]^{1/\rho}$$
$$= y \left(p_1^r + p_2^r \right)^{-1/r}$$

It is easy to see that $v(\mathbf{p}, y)$ is homogeneous of degree zero in prices and income, because for any t > 0,

$$v(t\mathbf{p}, ty) = ty((tp_1)^r + (tp_2)^r)^{-1/r}$$

$$= ty(t^r p_1^r + t^r p_2^r)^{-1/r}$$

$$= tyt^{-1}(p_1^r + p_2^r)^{-1/r}$$

$$= y(p_1^r + p_2^r)^{-1/r}$$

$$= v(\mathbf{p}, y)$$



To see that it is increasing in y and decreasing in \mathbf{p} , we obtain

$$\frac{\partial v(\mathbf{p}, y)}{\partial y} = (p_1^r + p_2^r)^{-1/r} > 0$$

$$\frac{\partial v(\mathbf{p}, y)}{\partial p_i} = -(p_1^r + p_2^r)^{(-1/r)-1} y p_i^{r-1} < 0, \quad i = 1, 2$$

To verify Roy's identity, we obtain

$$(-1)\left[\frac{\partial v(\mathbf{p}, y)/\partial p_i}{\partial v(\mathbf{p}, y)/\partial y}\right] = (-1)\frac{-(p_1^r + p_2^r)^{(-1/r)-1}yp_i^{r-1}}{(p_1^r + p_2^r)^{-1/r}}$$
$$= \frac{yp_i^{r-1}}{p_1^r + p_2^r} = x_i(\mathbf{p}, y), \quad i = 1, 2$$