2. The Envelope Theorem

$$V(\mathbf{a}) = \max_{\mathbf{x}\in R^n} f(\mathbf{x},\mathbf{a}) \text{ s.t. } g(\mathbf{x},\mathbf{a}) = 0 \quad \mathbf{x} = (x_1,\ldots,x_n)$$

$$\mathbf{a} = (a_1,\ldots,a_j)$$

The Envelope theorem states that for every $\mathbf{a} \in \mathrm{U}$,

$$\frac{\partial V(\mathbf{a})}{\partial a_j} = \left.\frac{\partial L}{\partial a_j}\right|_{\mathbf{x}(\mathbf{a}),\boldsymbol{\lambda}(\mathbf{a})} \qquad j = 1,\ldots,m$$

Proof

Form the Lagrangian for the maximisation problem

$$L \equiv f(\mathbf{x},\mathbf{a}) + \lambda[g(\mathbf{x},\mathbf{a})]$$

For every $\mathbf{a} \in \mathrm{U}$

$$\frac{\partial f(\mathbf{x}(\mathbf{a}),\mathbf{a})}{\partial x_i} + \lambda(\mathbf{a})\frac{\partial g(\mathbf{x}(\mathbf{a}),\mathbf{a})}{\partial x_i} = 0$$

$$g(\mathbf{x}(\mathbf{a}),\mathbf{a}) = 0 \qquad ①$$

If we evaluate this derivative at the point $(\mathbf{x}(\mathbf{a}),\lambda(\mathbf{a}))$

$$\left.\frac{\partial L}{\partial a_j}\right|_{\mathbf{x}(\mathbf{a}),\lambda(\mathbf{a})}=\frac{\partial f(\mathbf{x(a)},\mathbf{a})}{\partial a_j}+\lambda(\mathbf{a})\frac{\partial g(\mathbf{x(a)},\mathbf{a})}{\partial a_j} \qquad ②$$

We begin by directly differentiating $V(\mathbf{a})$ *with respect to* a_j*. Because* a_j *affects f directly and indirectly through its influence on each variable* $x_i(\mathbf{a})$

$$\frac{\partial V(\mathbf{a})}{\partial a_j}=\sum_{i=1}^{n}\left[\frac{\partial f(\mathbf{x}(\mathbf{a}),\mathbf{a})}{\partial x_i}\bullet\frac{\partial x_i(\mathbf{a})}{\partial a_j}\right]+\frac{\partial f(\mathbf{x}(\mathbf{a}),\mathbf{a})}{\partial a_j}$$

Go back to the first-order conditions, substituting into the bracketed term of the summation.

$$\frac{\partial V(\mathbf{a})}{\partial a_j} = -\lambda(\mathbf{a}) \sum_{i=1}^{n} \left[\frac{\partial g(\mathbf{x}(\mathbf{a}), \mathbf{a})}{\partial x_i} \bullet \frac{\partial x_i(\mathbf{a})}{\partial a_j} \right] + \frac{\partial f(\mathbf{x}(\mathbf{a}), \mathbf{a})}{\partial a_j} \qquad ③$$

Go back again to the first-order conditions ① and look at the second identity in the system.

$$\because g(\mathbf{x}(\mathbf{a}),\mathbf{a}) = 0$$

$$\Rightarrow \sum_{i=1}^{n}\left[\frac{\partial g(\mathbf{x}(\mathbf{a}),\mathbf{a})}{\partial x_i} \bullet \frac{\partial x_i(\mathbf{a})}{\partial a_j}\right] + \frac{\partial g(\mathbf{x}(\mathbf{a}),\mathbf{a})}{\partial a_j} = 0$$

$$\therefore \frac{\partial V(\mathbf{a})}{\partial a_j} = \left.\frac{\partial L}{\partial a_j}\right|_{\mathbf{x}(\mathbf{a}),\lambda(\mathbf{a})}$$