3.The Indirect Utility Function

The relationship among prices, income, and the maximised value of utility can be summarised as follows

$$v(\mathbf{p}, y) = \max_{\mathbf{x} \in R^n_+} u(\mathbf{x}) \quad \text{s.t.} \quad \mathbf{p} \cdot \mathbf{x} \le y$$

The maximum level of utility that can be achieved when facing prices $\mathbf{p}$ *and income* y *therefore will be that which is realised when* $\mathbf{x}(\mathbf{p}, y)$ *is chosen. Hence,*

$$v(\mathbf{p}, y) = u(\mathbf{x}(\mathbf{p}, y))$$

$$\mathbf{x}^* \equiv \mathbf{x}(\mathbf{p}, y)$$

Properties of the Indirect Utility Function

- *Homogeneous of degree zero in* $(\mathbf{p}, y)$
- *Strictly increasing in* y
- *Decreasing in* $\mathbf{p}$
- *Quasiconvex in* $(\mathbf{p}, y)$
- *Roy's identity*

$$x_i(\mathbf{p}^0, y^0) = -\frac{\partial v(\mathbf{p}^0, y^0)/\partial p_i}{\partial v(\mathbf{p}^0, y^0)/\partial y}, \quad i = 1, \ldots, n$$

Proof

$$\textit{Because } u(\cdot) \textit{ is strictly increasing}$$

$$v(\mathbf{p}, y) = \max_{\mathbf{x} \in \mathbb{R}^n_+} u(\mathbf{x}) \quad \text{s.t.} \quad \mathbf{p} \cdot \mathbf{x} = y$$

$$\textit{The Lagrangian is}$$

$$L(\mathbf{x}, \lambda) = u(\mathbf{x}) + \lambda(y - \mathbf{p} \cdot \mathbf{x})$$

$$\textit{Then there is a } \lambda^* \in \mathbb{R} \textit{ such that}$$

$$\frac{\partial L(\mathbf{x}^*, \lambda^*)}{\partial x_i} = \frac{\partial u(\mathbf{x}^*)}{\partial x_i} - \lambda^* p_i = 0, \quad i = 1, \ldots, n$$

According to the Envelope theorem

$$\frac{\partial v(\mathbf{p}, y)}{\partial y} = \frac{\partial L(\mathbf{x}^*, \lambda^*)}{\partial y} = \lambda^* > 0$$

$$\frac{\partial v(\mathbf{p}, y)}{\partial p_i} = \frac{\partial L(\mathbf{x}^*, \lambda^*)}{\partial p_i} = -\lambda^* x_i^* \leq 0$$

$$Let\ \ \mathbf{x}^0\ \ solves\ \ the\ \ maximisation\ \ problem\ when\ \ \mathbf{p}=\mathbf{p}^0$$

$$that\ \ is\ \ v(\mathbf{p}^0, y) = u(\mathbf{x}^0) \quad (\mathbf{x}^0 = \mathbf{x}(\mathbf{p}^0, y))$$

$$Set\ \ \mathbf{p}^0 \geq \mathbf{p}^1 \quad so\ (\mathbf{p}^0 - \mathbf{p}^1)\cdot \mathbf{x}^0 \geq 0 \qquad (since\ \ \mathbf{x}^0 \geq 0)$$

$$Hence\ \ \mathbf{p}^1 \cdot \mathbf{x}^0 \leq \mathbf{p}^0 \cdot \mathbf{x}^0 \leq y \quad so\ \ \mathbf{x}^0\ is\ \ feasible\ \ when\ \mathbf{p} = \mathbf{p}^1$$

$$So\ v(\mathbf{p}^1, y) \geq u(x^0) = v(\mathbf{p}^0, y), as\ desired$$

$$B^1 = \{\mathbf{x} \mid \mathbf{p}^1 \cdot \mathbf{x} \le y^1\} \quad (budget\, set\ available\, at\, (\mathbf{p}^1, y^1))$$

$$B^2 = \{\mathbf{x} \mid \mathbf{p}^2 \cdot \mathbf{x} \le y^2\} \quad (budget\, set\ available\, at\, (\mathbf{p}^2, y^2))$$

$$B^t = \{\mathbf{x} \mid \mathbf{p}^t \cdot \mathbf{x} \le y^t\} \quad (budget\, set\ available\, at\, (\mathbf{p}^t, y^t))$$

$$\mathbf{p}^t \equiv t\mathbf{p}^1 + (1-t)\mathbf{p}^2$$

$$y^t \equiv ty^1 + (1-t)y^2$$

$$\text{If } \mathbf{x} \in B^t \text{ then } \mathbf{x} \in B^1 \cup B^2 \text{ for } \forall\, t \in [0,1]$$

$$\text{Suppose there are some } \mathbf{x} \in B^t \text{ such that } \mathbf{x} \notin B^1 \text{ and } \mathbf{x} \notin B^2$$

$$\text{Then } \mathbf{p}^1 \cdot \mathbf{x} > y^1, \mathbf{p}^2 \cdot \mathbf{x} > y^2, \text{hence } \mathbf{p}^t \cdot \mathbf{x} > y^t$$

But it contradicts our original assumption.

$$\text{Therefore, if } \mathbf{x} \in B^t \text{ then } \mathbf{x} \in B^1 \cup B^2 \text{ for } \forall\, t \in [0,1]$$

$$\text{Hence } v(\mathbf{p}^t, y^t) \le \max\left[v(\mathbf{p}^1, y^1), v(\mathbf{p}^2, y^2)\right] \text{ for } \forall\, t \in [0,1]$$

$$v(\mathbf{p}, y) \text{ is quasiconvex in } (\mathbf{p}, y)$$

As for ***Roy's identity****,*
according to the Envelope theorem

$$\lambda^* = \partial v(\mathbf{p}, y) / \partial y > 0$$

$$\frac{\partial v(\mathbf{p}, y)}{\partial p_i} = \frac{\partial L(\mathbf{x}^*, \lambda^*)}{\partial p_i} = -\lambda^* x_i^*$$

Therefore

$$-\frac{\partial v(\mathbf{p}, y) / \partial p_i}{\partial v(\mathbf{p}, y) / \partial y} = -\frac{-\lambda^* x_i^*}{\lambda^*} = x_i^* \equiv x_i(\mathbf{p}, y)$$