4.CES Utility Function

CES utility function represents preferences that are strictly monotonic and strictly convex.

$$u(x_1, x_2) = \left(x_1{}^{\rho} + x_2{}^{\rho}\right)^{1/\rho} \quad 0 \neq \rho < 1$$

The consumer's problem is to find a non-negative consumption bundle solving

$$\max_{x_1, x_2} \left(x_1^{\rho} + x_2^{\rho}\right)^{1/\rho} \quad \text{s.t.} \quad p_1 x_1 + p_2 x_2 - y \leq 0$$

To solve this problem, we first form the associated Lagrangian

$$L(x_1, x_2, \lambda) \equiv \left(x_1^{\rho} + x_2^{\rho}\right)^{1/\rho} - \lambda(p_1 x_1 + p_2 x_2 - y)$$

Assuming an interior solution

$$\frac{\partial L}{\partial x_1} = \left(x_1^{\rho} + x_2^{\rho}\right)^{(1/\rho)-1} x_1^{\rho-1} - \lambda p_1 = 0$$

$$\frac{\partial L}{\partial x_2} = \left(x_1^{\rho} + x_2^{\rho}\right)^{(1/\rho)-1} x_2^{\rho-1} - \lambda p_2 = 0$$

$$\frac{\partial L}{\partial \lambda} = p_1 x_1 + p_2 x_2 - y = 0$$

Then

$$x_1 = x_2 \left(\frac{p_1}{p_2}\right)^{1/(\rho-1)}$$

$$y = p_1 x_1 + p_2 x_2$$

Solving for x_2 *and* x_1 *gives the solution value*

$$x_2 = \frac{p_2^{1/(\rho-1)} y}{p_1^{\rho/(\rho-1)} + p_2^{\rho/(\rho-1)}}$$

$$x_1 = \frac{p_1^{1/(\rho-1)} y}{p_1^{\rho/(\rho-1)} + p_2^{\rho/(\rho-1)}}$$

Let $r = \rho/(\rho-1)$

Then we can write the Marshallian demands as

$$x_1(\mathbf{p}, y) = \frac{p_1^{r-1} y}{p_1^r + p_2^r}$$

$$x_2(\mathbf{p}, y) = \frac{p_2^{r-1} y}{p_1^r + p_2^r}$$

By substituting the Marshallian demands back into the direct utility function, we get

$$v(\mathbf{p}, y) = \left[(x_1(\mathbf{p}, y))^{\rho} + (x_2(\mathbf{p}, y))^{\rho} \right]^{1/\rho}$$

$$= y\left(p_1^r + p_2^r\right)^{-1/r}$$

It is easy to see that v(**p***, y) is homogeneous of degree zero in prices and income, because for any t > 0,*

$$v(t\mathbf{p}, ty) = ty\left((tp_1)^r + (tp_2)^r\right)^{-1/r}$$

$$= ty\left(t^r p_1^r + t^r p_2^r\right)^{-1/r}$$

$$= tyt^{-1}\left(p_1^r + p_2^r\right)^{-1/r}$$

$$= y\left(p_1^r + p_2^r\right)^{-1/r}$$

$$= v(\mathbf{p}, y)$$

To see that it is increasing in y and decreasing in **p**, *we obtain*

$$\frac{\partial v(\mathbf{p}, y)}{\partial y} = \left(p_1^r + p_2^r\right)^{-1/r} > 0$$

$$\frac{\partial v(\mathbf{p}, y)}{\partial p_i} = -\left(p_1^r + p_2^r\right)^{(-1/r)-1} y p_i^{r-1} < 0, \quad i = 1,2$$

To verify Roy's identity, we obtain

$$(-1)\left[\frac{\partial v(\mathbf{p}, y)/\partial p_i}{\partial v(\mathbf{p}, y)/\partial y}\right] = (-1)\frac{-\left(p_1^r + p_2^r\right)^{(-1/r)-1} y p_i^{r-1}}{\left(p_1^r + p_2^r\right)^{-1/r}}$$

$$= \frac{y p_i^{r-1}}{p_1^r + p_2^r} = x_i(\mathbf{p}, y), \quad i = 1,2$$