5.The Expenditure Function

We define the expenditure function as the minimum-value function

$$e(\mathbf{p},u) \equiv \min_{\mathbf{x}\in \mathbb{R}_{+}^{n}} \mathbf{p}\cdot\mathbf{x} \quad \text{s.t.} \quad u(\mathbf{x}) \geq u$$

$$e(\mathbf{p},u) = \mathbf{p}\cdot\mathbf{x}^{h}(\mathbf{p},u)$$

$$\mathbf{x}^{*} \equiv \mathbf{x}^{h}(\mathbf{p},u)$$

(*compensated demand functions*)

Properties of the Expenditure Function

- *Increasing in* $\mathbf{p}$
- *Homogeneous of degree 1 in* $\mathbf{p}$
- *Concave in* $\mathbf{p}$
- *Shephard's lemma*

$$\frac{\partial e(\mathbf{p}^0, u^0)}{\partial p_i} = x_i^h(\mathbf{p}^0, u^0), \quad i = 1, \ldots, n$$

Proof

The Lagrangian for this problem is

$$L(\mathbf{x},\lambda)=\mathbf{p}\cdot\mathbf{x}-\lambda[u(\mathbf{x})-u]$$

So, by Lagrange's theorem, there is a λ^* *such that*

$$\frac{\partial L(\mathbf{x}^*,\lambda^*)}{\partial x_i}=p_i-\lambda^*\frac{\partial u(\mathbf{x}^*)}{\partial x_i}=0,\quad i=1,\ldots,n$$

By the Envelope theorem

$$\frac{\partial e(\mathbf{p},u)}{\partial u}=\frac{\partial L(\mathbf{x}^*,\lambda^*)}{\partial u}=\lambda^*>0$$

$$Let\ \mathbf{p} \equiv t\mathbf{p} + (1-t)\mathbf{p}\ \ for\ \forall\ t \in [0,1]$$

Then the expenditure function will be concave in $\mathbf{p}$ *if*

$$te(\mathbf{p}^1, u) + (1-t)e(\mathbf{p}^2, u) \le e(\mathbf{p}^t, u)$$

$$Suppose\ \mathbf{x}^1 \equiv \mathbf{x}^h(\mathbf{p}^1, u)$$

$$\mathbf{x}^2 \equiv \mathbf{x}^h(\mathbf{p}^2, u)$$

$$\mathbf{x}^* \equiv \mathbf{x}^h(\mathbf{p}^*, u)$$

$$Then\ \mathbf{p}^1 \cdot \mathbf{x}^1 \le \mathbf{p}^1 \cdot \mathbf{x},\ \mathbf{p}^2 \cdot \mathbf{x}^2 \le \mathbf{p}^2 \cdot \mathbf{x}$$

$$So\ \ \mathbf{p}^1 \cdot \mathbf{x}^1 \le \mathbf{p}^1 \cdot \mathbf{x}^*,\ \mathbf{p}^2 \cdot \mathbf{x}^2 \le \mathbf{p}^2 \cdot \mathbf{x}^*$$

$$t\mathbf{p}^1 \cdot \mathbf{x}^1 + (1-t)\mathbf{p}^2 \cdot \mathbf{x}^2 \le \mathbf{p}^t \cdot \mathbf{x}^*$$

It tells us that

$$te(\mathbf{p}^1, u) + (1-t)e(\mathbf{p}^2, u) \le e(\mathbf{p}^t, u), \forall\, t \in [0,1]$$

By the Envelope theorem

$$\frac{\partial e(\mathbf{p}, u)}{\partial p_i} = \frac{\partial L(\mathbf{x}^*, \lambda^*)}{\partial p_i} = x_i^* \equiv x_i^h(\mathbf{p}, u)$$