6.The CES Case

Suppose the direct utility function is again the CES form.

We want to derive the corresponding expenditure function in this case.

Because preferences are monotonic, we can formulate the expenditure minimisation problem

$$\min_{x_1, x_2} p_1 x_1 + p_2 x_2 \quad \text{s.t.} \quad \left(x_1^{\rho} + x_2^{\rho}\right)^{1/\rho} - u = 0, \quad x_1 \geq 0, x_2 \geq 0$$

and its Lagrangian

$$L(x_1, x_2, \lambda) = p_1 x_1 + p_2 x_2 - \lambda\left[\left(x_1^{\rho} + x_2^{\rho}\right)^{1/\rho} - u\right]$$

Assuming an interior solution in both goods

$$\frac{\partial L}{\partial x_1} = p_1 - \lambda\left(x_1^{\rho} + x_2^{\rho}\right)^{(1-\rho)-1} x_1^{\rho-1} = 0$$

$$\frac{\partial L}{\partial x_2} = p_2 - \lambda\left(x_1^{\rho} + x_2^{\rho}\right)^{(1-\rho)-1} x_2^{\rho-1} = 0$$

$$\frac{\partial L}{\partial \lambda} = \left(x_1^{\rho} + x_2^{\rho}\right)^{1/\rho} - u = 0$$

By eliminating λ

$$x_1 = x_2\left(\frac{p_1}{p_2}\right)^{1/(\rho-1)}$$

$$u = \left(x_1^{\rho} + x_2^{\rho}\right)^{1/\rho}$$

Then

$$u=\left[x_2^{\rho}\left(\frac{p_1}{p_2}\right)^{\rho/(\rho-1)}+x_2^{\rho}\right]^{1/\rho}=x_2\left[\left(\frac{p_1}{p_2}\right)^{\rho/(\rho-1)}\times 1\right]^{1/\rho}$$

Solving for x_2 *and* x_1, *we obtain*

$$x_2=u\left[\left(\frac{p_1}{p_2}\right)^{\rho/(\rho-1)}+1\right]^{-1/\rho}=u\left[p_1^{\rho/(\rho-1)}+p_2^{\rho/(\rho-1)}\right]^{-1/\rho}p_2^{1/(\rho-1)}$$

$$=u\left(p_1^{r}+p_2^{r}\right)^{(1/r)-1}p_2^{r-1}$$

$$x_1=up_1^{1/(\rho-1)}p_2^{-1/(\rho-1)}\left(p_1^{r}+p_2^{r}\right)^{(1/r)-1}p_2^{r-1}$$

$$=u\left(p_1^{r}+p_2^{r}\right)^{(1/r)-1}p_1^{r-1}$$

These are the Hicksian demands, so we obtain

$$x_1^h(\mathbf{p},u)=u\left(p_1^r+p_2^r\right)^{(1/r)-1}p_1^{r-1}$$

$$x_2^h(\mathbf{p},u)=u\left(p_1^r+p_2^r\right)^{(1/r)-1}p_2^{r-1}$$

Then the expenditure function

$$\begin{aligned}
e(\mathbf{p},u) &= p_1x_1^h(\mathbf{p},u)+p_2x_2^h(\mathbf{p},u)\\
&=up_1\left(p_1^r+p_2^r\right)^{(1/r)-1}p_1^{r-1}+up_2\left(p_1^r+p_2^r\right)^{(1/r)-1}p_2^{r-1}\\
&=u\left(p_1^r+p_2^r\right)\left(p_1^r+p_2^r\right)^{(1/r)-1}\\
&=u\left(p_1^r+p_2^r\right)^{1/r}
\end{aligned}$$