7.Relations Between the Two

Though the indirect utility function and the expenditure function are conceptually distinct, there is obviously a close relationship between them.

Let $v(\mathbf{p}, y)$ *and* $e(\mathbf{p}, u)$ *be the indirect utility function and expenditure function for some consumer whose utility function is continuous and strictly increasing.*

Then for all $\mathbf{p} \gg \mathbf{0}$, $y \geq 0$, *and* $u \in \mathrm{U}$

$$e(\mathbf{p}, v(\mathbf{p}, y)) = y$$
$$v(\mathbf{p}, e(\mathbf{p}, u)) = u$$

As the CES direct utility function gives the indirect utility function

$$v(\mathbf{p}, y) = y\left(p_1^r + p_2^r\right)^{-1/r} \text{ for any } \mathbf{p} \text{ and income level } y$$

Therefore, for an income level y *equal to* $e(\mathbf{p}, u)$ *dollars*

$$v(\mathbf{p}, e(\mathbf{p}, u)) = e(\mathbf{p}, u)\left(p_1^r + p_2^r\right)^{-1/r}$$

Next, for any $\mathbf{p}$ *and* u

$$v(\mathbf{p}, e(\mathbf{p}, u)) = u$$

Then $e(\mathbf{p}, u)\left(p_1^r + p_2^r\right)^{-1/r} = u$

Now we get the expression for the expenditure function

$$e(\mathbf{p},u) = u\left(p_1^r + p_2^r\right)^{1/r}$$

This is the same expression for the expenditure function obtained by directly solving the consumer's expenditure-minimisation problem.

Suppose, instead, we begin with knowledge of the expenditure function and want to derive the indirect utility function.

For the CES direct utility function

$$e(\mathbf{p},u)=u\left(p_1^r + p_2^r\right)^{1/r}$$

Then for utility level $v(\mathbf{p}, y)$, we will have

$$e(\mathbf{p},v(\mathbf{p},y))=v(\mathbf{p},y)\left(p_1^r + p_2^r\right)^{1/r}$$

Next, for any $\mathbf{p}$ and y

$$e(\mathbf{p},v(\mathbf{p},y))=y$$

Then $(\mathbf{p},y)\left(p_1^r + p_2^r\right)^{1/r} = y$

Now we get the expression for the expenditure function

$$v(\mathbf{p}, y) = y\left(p_1^r + p_2^r\right)^{-1/r}$$

This is what we obtained by directly solving the consumer's utility-maximisation problem.

We can pursue this relationship between utility maximisation and expenditure minimisation a bit further by shifting our attention to the respective solutions to these two problems, the Marshallian and Hicksian demand functions.