8. Duality Between Marshallian and Hicksian Demand Functions

We have the following relations between the Hicksian and Marshallian demand functions for $\mathbf{p} >> \mathbf{0}$, $y \geq 0$, $u \in \mathrm{U}$, *and* $i = 1,\ldots,n$

$$1.\ x_i(\mathbf{p}, y) = x_i^h(\mathbf{p}, v(\mathbf{p}, y))$$

$$2.\ x_i^h(\mathbf{p}, u) = x_i(\mathbf{p}, e(\mathbf{p}, u))$$

Example 1:

Let us confirm this theorem for a CES consumer. The Hicksian demands are

$$x_i^h(\mathbf{p},u)=u\left(p_1^r+p_2^r\right)^{(1/r)-1}p_i^{r-1} \quad i=1,2 \qquad ①$$

The indirect utility function is

$$v(\mathbf{p},y)=y\left(p_1^r+p_2^r\right)^{-1/r} \qquad ②$$

Substituting from ② for u in ① gives

$$
\begin{aligned}
x_i^h(\mathbf{p}, v(\mathbf{p}, y)) &= v(\mathbf{p}, y)\left(p_1^r + p_2^r\right)^{(1/r)-1} p_i^{r-1} \\
&= y\left(p_1^r + p_2^r\right)^{-1/r}\left(p_1^r + p_2^r\right)^{(1/r)-1} p_i^{r-1} \\
&= y p_i^{r\text{-}1}\left(p_1^r + p_2^r\right)^{-1} \\
&= \frac{y p_i^{r-1}}{p_1^r + p_2^r}, \quad i = 1,2
\end{aligned}
$$

Example 2:

The Marshallian demands are

$$x_i(\mathbf{p}, y) = \frac{y p_i^{r-1}}{p_1^r + p_2^r}, \quad i = 1,2 \qquad ③$$

The expenditure function is

$$e(\mathbf{p}, u) = u\left(p_1^r + p_2^r\right)^{1/r} \qquad ④$$

Substituting from ④ into ③ for y yields

$$x_i(\mathbf{p}, e(\mathbf{p}, u)) = \frac{e(\mathbf{p}, u) p_i^{r-1}}{p_1^r + p_2^r}$$

$$= u\left(p_1^r + p_2^r\right)^{1/r} \frac{p_i^{r-1}}{p_1^r + p_2^r}$$

$$= u p_i^{r\text{-}1}\left(p_1^r + p_2^r\right)^{(1/r)-1}, \ i = 1,2$$