9. The Slutsky Equation

Let $\mathbf{x}(\mathbf{p}, y)$ *be the consumer's Marshallian demand system. Let* u^* *be the level of utility the consumer achieves at prices* $\mathbf{p}$ *and income y.*

Then

$$\underbrace{\frac{\partial x_i(\mathbf{p}, y)}{\partial p_j}}_{TE} = \underbrace{\frac{\partial x_i^h(\mathbf{p}, u^*)}{\partial p_j}}_{SE} \underbrace{- x_j(\mathbf{p}, y)\frac{\partial x_i(\mathbf{p}, y)}{\partial y}}_{IE} \quad , \quad i,j = 1, \ldots, n$$

$$\left(u^* = v(\mathbf{p}, y)\right)$$

Proof

For any prices and level of utility u^*

$$x_i^h(\mathbf{p}, u^*) = x_i(\mathbf{p}, e(\mathbf{p}, u^*))$$

Because this holds for all $\mathbf{p} >> \mathbf{0}$*, we can differentiate both sides with respect to* p_j *and the equality is preserved. The Hicksian demand on the left-hand side, the Marshallian demand on the right-hand side*

$$\frac{\partial x_i^h(\mathbf{p}, u^*)}{\partial p_j} = \frac{\partial x_i(\mathbf{p}, e(\mathbf{p}, u^*))}{\partial p_j} + \frac{\partial x_i(\mathbf{p}, e(\mathbf{p}, u^*))}{\partial y} \frac{\partial e(\mathbf{p}, u^*)}{\partial p_j} \quad ①$$

We know that the minimum expenditure at prices **p** *and maximum utility that can be achieved at prices* **p** *and income y is equal to income y. Therefore*

$$e(\mathbf{p}, u^*) = e(\mathbf{p}, v(\mathbf{p}, y)) = y \qquad ②$$

The partial with respect to p_j *of the expenditure function in ① is just the Hicksian demand for good j at utility* u^*. *Because* $u^* = v(\mathbf{p}, y)$, *this must also be the Hicksian demand for good j at utility v(***p***, y)*

$$\frac{\partial e(\mathbf{p}, u^*)}{\partial p_j} = x_j^h(\mathbf{p}, u^*) = x_j^h(\mathbf{p}, v(\mathbf{p}, y)) = x_j(\mathbf{p}, y) \quad ③$$

(*by Shaphard's lemma*)

To complete the proof, substitute from ② and ③ into ① to obtain

$$\frac{\partial x_i^h(\mathbf{p}, u^*)}{\partial p_j} = \frac{\partial x_i(\mathbf{p}, y)}{\partial p_j} + \frac{\partial x_i(\mathbf{p}, y)}{\partial y} x_j(\mathbf{p}, y)$$

With a bit of rearranging, we have what we wanted to show

$$\frac{\partial x_i(\mathbf{p}, y)}{\partial p_j} = \frac{\partial x_i^h(\mathbf{p}, u^*)}{\partial p_j} - x_i(\mathbf{p}, y)\frac{\partial x_i(\mathbf{p}, y)}{\partial y} \quad i,j = 1, \dots, n$$

Slutsky equations provide neat analytical expressions for substitution and income effects. They also give us an 'accounting framework' , detailing how these must combine to explain any total effect of a given price change. Yet by themselves, the Slutsky relations do not answer any of the questions we set out to address.